Supplementary Mid-Term Exam — ISI Bangalore

November 2019

Algebra I — 100 Points

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) $\mathbb{Z}_n$ = the set of integers modulo n, $(\mathbb{Z}_n)^\times$ = the set of those integers in $\mathbb{Z}_n$ that are coprime to n.

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1. [10 points] How many different equivalence relations can be defined on a set of 5 elements?

2. [15 points] Find the gcd of $a = 1761$, $b = 1567$ and write it in the form $ax + by$ for suitable integers x, y.

3. [15 points] Let $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +)$ denote the additive groups of integers, rational numbers and real numbers respectively. Let $(G, \cdot)$ denote the multiplicative subgroup of $(\mathbb{C}^*, \cdot)$ consisting of the various roots of unity, i.e., $G = \{z \in \mathbb{C}^* \mid z^n = 1 \text{ for some } n > 0\}$. Prove that none of these 4 groups are isomorphic to each other.

4. [15 points] Prove or disprove using an example: If G is a group, then the function $f : G \to G$ given by $f(x) = x^{-1}$ is an isomorphism.

5. [15 points] Prove that the multiplicative group $((\mathbb{Z}_{14})^\times, \cdot)$ is isomorphic to the additive group $(\mathbb{Z}_6, +)$.

6. [15 points] Prove that S_n is generated by its 2-cycles.

7. [15 points] Suppose x, y are elements of a group G with $xy = y^a x$ for some integer $a > 0$. If $x^b = e$ for integer $b > 0$, prove that $y^{a^b - 1} = e$.